

Hamiltonian vs Lagrangian Embedding of a Massive Spin-one Theory Involving 2-form Field

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Abstract

We consider the Hamiltonian and Lagrangian embedding of a first-order, massive spin-one, gauge non-invariant theory involving anti-symmetric tensor field. We apply the BFV-BRST generalised canonical approach to convert the model to a first class system and construct nil-potent BFV-BRST charge and an unitarising Hamiltonian. The canonical analysis of the Stückelberg formulation of this model is presented. We bring out the contrasting feature in the constraint structure, specifically with respect to the reducibility aspect, of the Hamiltonian and the Lagrangian embedded model. We show that to obtain manifestly covariant Stückelberg Lagrangian from the BFV embedded Hamiltonian, phase space has to be further enlarged and show how the reducible gauge structure emerges in the embedded model.

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Introduction

The problem of quantisation of gauge systems is handled either in the covariant BRST framework [1] or in the canonical framework [2,3]. In the BRST scheme one constructs a nil-potent BRST charge (Q_{BRST}) which generates the global symmetry transformations of the gauge fixed action and the physical states has to satisfy the condition $Q_{BRST}|\psi\rangle = 0$. In the canonical framework, after fixing the gauge, in the reduced phase space, Poisson brackets are replaced by Dirac brackets. For systems with second class constraints one first replaces the Poisson brackets with Dirac brackets, and pass over to quantum commutation relations and treat the constraints as operator equations. But in many cases, this procedure of Dirac runs into problems related to the factor ordering ambiguity. To overcome the problems associated with the canonical method, generalised canonical quantisation methods have been developed [4]. In these generalised canonical methods, the second class system is first converted to a first class system and then schemes of quantisation for gauge theories can be fruitfully applied. This systematic conversion of the Hamiltonian and the second class constraints to a gauge invariant Hamiltonian and first class constraints is called Hamiltonian embedding. Even within the Lagrangian formulation, one can convert second class system to a gauge invariant theory. This Lagrangian embedding is achieved by introducing Stückelberg fields with compensating transformations resulting in the gauge invariant Lagrangian starting from a gauge non-invariant Lagrangian. In all known examples, these two procedures are found to be equivalent.

The basic approach in Hamiltonian embedding is to enlarge the phase space by introducing new variables using which the constraints and Hamiltonian are modified to obtain a first class theory. In one of these schemes, known as Batalin-Fradkin-Tyutin method, one first enlarges the phase space using variables with same Grassman parity as that of the constraint and modifies the second class constraints and Hamiltonian into a first class one. Thus after embedding the model with second class algebra into a model with involutive algebra (i.e., Poisson brackets between the constraints vanish, at least weakly), one quantises the first class, embedded model using well established procedures. In another variation of generalised canonical scheme known as Batalin-Fradkin-Vilkovisky-BRST (BFV-BRST) quantisation, the phase space is enlarged by introducing new phase space variables, Lagrange multipliers and their momenta (all having the same Grassman parity as that of the second class constraints) and also canonically conjugate fields (this include ghost fields) with opposite Grassman parity to that of the constraints. In this space, a gauge invariant Hamiltonian and a nil-potent BFV-BRST charge are obtained. Using the BFV-BRST nil-potent charge, the unitarising Hamiltonian is constructed. The unitarity and the gauge independence are guaranteed in this scheme of quantisation.

Generally, the new phase space variables (with same Grassman parity as that of the constraints) are identified with the Stückelberg fields in the Lagrangian formulation. Earlier the Hamiltonian and Lagrangian embedding analysis have been made for Proca model [5] and $2 + 1$ dimensional self-dual model [6] and the differences in the nature of

the constraint structure of the embedded models were clarified.

In this paper we study the Hamiltonian and Lagrangian embeddings of a new massive spin-one theory described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}G_\mu G^\mu + \frac{1}{2m}\epsilon_{\mu\nu\lambda\sigma}H^{\mu\nu}\partial^\lambda G^\sigma. \quad (1)$$

This first-order formulation is different from the first-order formulations of other well known models like Proca model, massive Kalb-Ramond or Duffin-Kemmer-Petiau formulation by being the first-order formulation of both Proca theory and massive Kalb-Ramond model. The interest in the model is also due to its equivalence to topologically massive $B\wedge F$ theory [7]. Motivation for the study of Hamiltonian and Lagrangian embeddings of the above model (1), is due to certain novel features present here in relation to reducibility, which are not shared by the other models studied earlier. We employ the BFV-BRST procedure to convert this (1) second class system to a first class Hamiltonian system and also construct the nil-potent BFV-BRST charge and the unitarising Hamiltonian. The first class constraints of the embedded system are found to be *irreducible*. We then start from the Lagrangian embedded model, which is the Stückelberg formulation and present its canonical analysis. The constraint structure here is *reducible*. This difference in the reducibility aspect of the constraints between the Hamiltonian and Lagrangian embedded model is not present in the earlier models studied. But as found in the earlier models, the difference in the nature of the constraints modified, by these two embedding procedures is observed here also. We also find that from the phase space path integral of the BFV embedded model, the Stückelberg Lagrangian obtained is not manifestly covariant. We show, by further enlarging the BFV-phase space, manifestly covariant Lagrangian is obtained. This enlargement also reconciles the difference in the reducibility nature of the constraints in embedded versions obtained by two procedures.

Recently the BFV-BRST procedure has been applied to few systems like the abelian Proca model [8] and the massive super particle [9]. The present work provides yet another illustration of this procedure, which by itself is of intrinsic interest.

This paper is organised in the following way. In section I, we construct the BFV-BRST nil-potent charge and unitarising Hamiltonian. In section II, we present the Hamiltonian analysis of the Lagrangian embedded model. In section III, we first point out the differences in the Hamiltonian embedded model and the Lagrangian embedded model. Then using the phase space path integral method we study the equivalence of BFV-BRST embedded model and Stückelberg formulation. Here we bring out the necessity of introducing the new phase space variables to get the manifestly covariant Stückelberg Lagrangian and also clarify how the reducible nature comes in the embedded model. We conclude with discussion in section IV.

We work with $g_{\mu\nu} = (1, -1, -1, -1)$ and $\epsilon_{0123} = 1$.

I. BFV-BRST Hamiltonian Embedding

In this section, by following Hamiltonian embedding procedure of BFV, we systematically convert all the second class constraints of the first-order formulation to first

class ones and construct a nil-potent charge. Using the BFV-BRST nil-potent charge, we construct a unitarising Hamiltonian.

We start with the Lagrangian, with the last term expressed in a symmetric form as,

$$\mathcal{L} = -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}G_\mu G^\mu + \frac{1}{4m}\epsilon_{\mu\nu\lambda\sigma}H^{\mu\nu}\partial^\lambda G^\sigma - \frac{1}{4m}\epsilon_{\mu\nu\lambda\sigma}\partial^\mu H^{\nu\lambda}G^\sigma. \quad (2)$$

The primary constraints following from the above Lagrangian are,

$$\Pi_0 \approx 0, \quad \Pi_{0i} \approx 0, \quad (3)$$

$$\Omega_i \equiv \left(\Pi_i - \frac{1}{4m}\epsilon_{0ijk}H^{jk} \right) \approx 0, \quad (4)$$

$$\text{and } \Lambda_{ij} \equiv \left(\Pi_{ij} + \frac{1}{2m}\epsilon_{0ijk}G^k \right) \approx 0. \quad (5)$$

and the Hamiltonian density following from the above Lagrangian (2) is,

$$\begin{aligned} H_T = & \frac{1}{4}H_{ij}H^{ij} - \frac{1}{2}G_i G^i + H_{0i} \left(\frac{1}{2}H^{0i} - \frac{1}{m}\epsilon^{0ijk}\partial_j G_k \right) \\ & - \frac{1}{2}G_0 \left(G^0 + \frac{1}{m}\epsilon^{0ijk}\partial_i H_{jk} \right) - (\partial^i G_i)\Pi_0 + (\partial^j H_{ij})\Pi^{0i}. \end{aligned} \quad (6)$$

The persistence of the primary constraints leads to the following secondary constraints,

$$\begin{aligned} \Lambda & \equiv \left(G_0 + \frac{1}{2m}\epsilon_{0ijk}\partial^i H^{jk} \right) \approx 0, \\ \Lambda_i & \equiv \left(-H_{0i} + \frac{1}{m}\epsilon_{0ijk}\partial^j G^k \right) \approx 0. \end{aligned} \quad (7)$$

It can be easily seen from their Poisson brackets that all the constraints are second class as expected for a theory without any gauge invariance. Note that the constraints Ω_i and Λ_{ij} are due to the symplectic structure of the Lagrangian (2). Following Faddeev and Jackiw [10], the symplectic conditions, which are not true constraints, are implemented strongly leading to the modified bracket,

$$\{G_i(\vec{x}), H_{jk}(\vec{y})\} = -m\epsilon_{0ijk}\delta(\vec{x} - \vec{y}). \quad (8)$$

Consequently, Ω_i and Λ_{ij} are implemented strongly.

Following the generalised Hamiltonian method, we enlarge the phase space by introducing canonically conjugate pair of ghost fields $C(x)_\alpha$ and $P(x)^\alpha$ corresponding to each constraint satisfying,

$$\begin{aligned} \{C(x)_\alpha, P(y)^\beta\} &= i\hbar\delta_\alpha^\beta\delta(x-y), \\ gh(C_\alpha) &= -gh(P^\alpha) = 1, \quad \mathcal{E}(C_\alpha) = \mathcal{E}(P_\alpha) = 1, \quad \alpha = 1, 2, 3, 4. \end{aligned} \quad (9)$$

where gh and \mathcal{E} stand for the ghost number and the Grassman parity respectively and $\{, \}$ stands for the graded commutator¹.

Now we define the operators,

$$\begin{aligned}\Omega &= \tilde{T}_\alpha C^\alpha + \Sigma_{n=1}^\infty \frac{1}{(n+1)!n!} P_{\alpha_1} \dots P_{\alpha n} U_{\beta_{n+1}}^{\alpha_1} U_{\beta_n \dots \beta_1}^{\alpha_2 \dots \alpha n} C^{\beta_1 \beta_2 \dots \beta_{n+1}} \\ \Omega^\alpha &= V_\beta^\alpha C^\beta + \Sigma_{n=1}^\infty \frac{1}{(n+1)!n!} P_{\alpha_1} \dots P_{\alpha n} V_{\beta_{n+1}}^{\alpha_1} V_{\beta_n \dots \beta_1}^{\alpha_2 \dots \alpha n} C^{\beta_1 \beta_2 \dots \beta_{n+1}},\end{aligned}\quad (10)$$

satisfying the conditions,

$$\begin{aligned}\{\Omega(x), \Omega(y)\} &= \int dx dy \Omega^\alpha \omega_{\alpha\beta} \Omega^\beta \\ \{\Omega^\alpha, \Omega^\beta\} &= 0, \quad \{\Omega^\alpha, \Omega\} = 0.\end{aligned}\quad (11)$$

where \tilde{T}_α for $\alpha = 1, 2, 3, 4$ are the original second class constraints Π_0 , Π_{0i} , Λ , and Λ_i (3,7) respectively and $\omega_{\alpha\beta}$ is an arbitrary, invertable anti-symmetric matrix. In our case we find

$$\Omega = \Pi_0 C^1 + \Pi_{0i} C^{2i} + \Lambda C^3 + \Lambda_i C^{4i} \quad (12)$$

$$\begin{aligned}\Omega_1 &= \frac{1}{\sqrt{2}}(C_1 - C_3), & \Omega_2 &= \frac{1}{\sqrt{2}}(C_2^i + C_4^i), \\ \Omega_3 &= \frac{1}{\sqrt{2}}(C_1 + C_3), & \Omega_4 &= \frac{1}{\sqrt{2}}(C_2^i - C_4^i),\end{aligned}\quad (13)$$

satisfying the above conditions (11).

We now further enlarge the phase space by introducing the variables Π_α , α , p_i , and q_i obeying

$$\begin{aligned}\{\alpha(x), \Pi(y)_\alpha\} &= i\hbar \delta(x-y) \\ \{q(x)_i, p(y)^j\} &= i\hbar \delta_i^j \delta(x-y), \\ gh(\alpha) &= gh(\Pi_\alpha) = gh(p_i) = gh(q_i) = 0, \\ \mathcal{E}(\alpha) &= \mathcal{E}(\Pi_\alpha) = \mathcal{E}(p_i) = \mathcal{E}(q_i) = 0,\end{aligned}\quad (14)$$

and define a nil-potent operator

$$\begin{aligned}\Omega' &= \Omega - \Omega_1 \Pi_\alpha - \Omega_2^i p_i + \Omega_3 \alpha + \Omega_4^i q_i \\ &= T_\alpha C^\alpha, \\ \text{and} \quad (\Omega')^2 &= \frac{1}{2} \{\Omega', \Omega'\} = 0.\end{aligned}\quad (15)$$

¹Following the BFV-BRST construction we replace all the Poisson brackets between the variables of the original second class model with commutators.

Here

$$\begin{aligned} T_1 &= \Pi_0 - \frac{1}{\sqrt{2}}(\pi_\alpha - \alpha), & T_2^i &= \Pi^{0i} - \frac{1}{\sqrt{2}}(p^i + q^i), \\ T_3 &= \Lambda + \frac{1}{\sqrt{2}}(\pi_\alpha + \alpha), & T_4^i &= \Lambda^i - \frac{1}{\sqrt{2}}(p^i - q^i), \end{aligned} \quad (16)$$

are the modified constraints that are in strong involution. Note that these first class constraints are linearly independent which can be seen by the degrees of freedom count.

To construct a first class Hamiltonian that is in strong involution with T_α , we first define,

$$\begin{aligned} \bar{\Omega}_1 &= \frac{1}{\sqrt{2}}(P_1 - P_3), & \bar{\Omega}_2^i &= \frac{1}{\sqrt{2}}(P_2^i + P_4^i), \\ \bar{\Omega}_3 &= \frac{1}{\sqrt{2}}(P_1 + P_3), & \bar{\Omega}_4^i &= \frac{1}{\sqrt{2}}(P_2^i - P_4^i), \end{aligned} \quad (17)$$

satisfying the conditions,

$$\begin{aligned} \{\Omega(x)_\alpha, \bar{\Omega}(y)^\beta\} &= i\hbar\delta_\alpha^\beta\delta(x-y), \\ \bar{\Omega}_\alpha\Omega^\alpha &= P_\alpha C^\alpha. \end{aligned} \quad (18)$$

Using these operators, one defines the first class Hamiltonian,

$$\begin{aligned} \mathcal{H} &= H_T + \frac{1}{(i\hbar)^2} \int dx \{H_T, \{\Omega, \bar{\Omega}^\alpha\}\} \Phi_\alpha \\ &+ \frac{1}{(i\hbar)^4} \int dx dy \{\{H_T, \{\Omega, \bar{\Omega}^\alpha\}\}, \{\Omega, \bar{\Omega}^\beta\}\} \Phi_\alpha \Phi_\beta + \dots, \end{aligned} \quad (19)$$

where Φ_α is α, Π_α, p_i , and q_i for $\alpha = 1, 2, 3$ and 4 respectively and the total Hamiltonian H_T is given in (6). We see that all the higher order terms in the above series vanish.

Using (6,12,17,18) in (19), we get

$$\begin{aligned} \mathcal{H} &= H_T - \Lambda\Pi_\theta + \partial^i\Pi_0\partial_i\theta - \Lambda_i\tilde{\Pi}^i + (\partial^j\partial_j\Pi_{0i} - \partial^j\partial_i\Pi_{0j})B^i \\ &- \frac{1}{2}(\Pi_\theta)^2 + \frac{1}{2}\partial^i\theta\partial_i\theta + \frac{1}{2}\tilde{\Pi}_i\tilde{\Pi}^i - \frac{1}{4}B_{ij}B^{ij}, \end{aligned} \quad (20)$$

where we have used the definitions,

$$\begin{aligned} \frac{1}{\sqrt{2}}(\alpha - \Pi_\alpha) &= \theta, & \frac{1}{\sqrt{2}}(\alpha + \Pi_\alpha) &= \Pi_\theta, \\ \frac{1}{\sqrt{2}}(q_i - p_i) &= B_i, & \frac{1}{\sqrt{2}}(q_i + p_i) &= \tilde{\Pi}_i, \end{aligned} \quad (21)$$

and $B_{ij} = (\partial_i B_j - \partial_j B_i)$. From (21) we notice that $\{\theta(x), \Pi(y)_\theta\} = i\hbar\delta(x-y)$ and $\{B(x)_i, \tilde{\Pi}(y)^j\} = i\hbar\delta_i^j\delta(x-y)$. Thus, we can re-express the involutive constraints (16) as

$$\begin{aligned}
T_1 &= \Pi_0 + \theta, & T_2^i &= \Pi^{0i} + B^i, \\
T_3 &= \Lambda + \Pi_\theta, & T_4^i &= \Lambda^i - \tilde{\Pi}^i.
\end{aligned} \tag{22}$$

Now we have the modified constraints T_α , Hamiltonian \mathcal{H} and the nil-potent operator Ω' satisfying

$$\begin{aligned}
\{T_\alpha, T_\beta\} &= 0, \\
\{T_\alpha, \mathcal{H}\} &= 0, \\
\text{and } \{\Omega', \mathcal{H}\} &= 0.
\end{aligned} \tag{23}$$

Now we further enlarge the phase space by introducing (dynamical) Lagrange multipliers λ_α and their conjugate momenta Π_λ^α corresponding to each of these first class constraints T_α satisfying,

$$\begin{aligned}
\{\lambda(x)_\alpha, \Pi(y)_\lambda^\beta\} &= i\hbar\delta_\alpha^\beta\delta(x-y), \\
gh(\lambda_\alpha) &= gh(\Pi_{\lambda_\alpha}) = 0, \quad \mathcal{E}(\lambda_\alpha) = \mathcal{E}(\Pi_{\lambda_\alpha}) = 0.
\end{aligned} \tag{24}$$

We also introduce the hermitian ghost-anti ghost pairs,

$$\{g(x)_\alpha, \bar{g}(y)^\beta\} = i\hbar\delta_\alpha^\beta\delta(x-y), \quad gh(g_\alpha) = -gh(\bar{g}_\alpha) = 1, \quad \mathcal{E}(g_\alpha) = \mathcal{E}(\bar{g}_\alpha) = 1 \tag{25}$$

and define the BFV-BRST operator

$$Q = \Omega' + \int dx \Pi_\lambda^\alpha g_\alpha \tag{26}$$

which is nil-potent ($Q^2 = 0$).

The gauge fixing Fermion is defined as,

$$\Psi = \int dx [P_\alpha \lambda^\alpha + \bar{g}^\alpha \chi_\alpha], \quad gh(\Psi) = -1, \tag{27}$$

where χ_α are the gauge fixing conditions to be fixed so that $\det\{T_\alpha, \chi_\beta\} \neq 0$. Using this Ψ , we construct the unitarising Hamiltonian

$$\mathcal{H}_u = \mathcal{H} + \frac{1}{i\hbar} \{\Psi, Q\}. \tag{28}$$

This completes the application of the BFV-BRST procedure to (2).

II. Lagrangian Embedding

In this section we present the Hamiltonian and the constraints the embedded Lagrangian. We start with the Stückelberg Lagrangian corresponding to (2) is,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(H_{\mu\nu} - B_{\mu\nu})(H^{\mu\nu} - B^{\mu\nu}) + \frac{1}{2}(G_\mu + \partial_\mu\theta)(G^\mu + \partial^\mu\theta) \\
&+ \frac{1}{4m}\epsilon_{\mu\nu\lambda\sigma}H^{\mu\nu}\partial^\lambda G^\sigma - \frac{1}{4m}\epsilon_{\mu\nu\lambda\sigma}\partial^\mu H^{\nu\lambda}G^\sigma,
\end{aligned} \tag{29}$$

where $B_{\mu\nu} = (\partial_\mu B_\nu - \partial_\nu B_\mu)$. The above Lagrangian is invariant under

$$\begin{aligned}\delta H_{\mu\nu} &= (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu), \quad \delta B_\mu = \Lambda_\mu + \partial_\mu \alpha, \\ \delta G_\mu &= \partial_\mu \lambda, \quad \delta \theta = \lambda.\end{aligned}\tag{30}$$

The Lagrangian is invariant under the variation of the gauge parameter Λ_μ ($\delta \Lambda_\mu = \partial_\mu \rho$), making the system reducible.

Next the canonical analysis of the gauge invariant theory described by the above Stückelberg Lagrangian is carried out to compare with the Hamiltonian and constraint structure of the embedded model.

The primary constraints following from the Lagrangian (29) are,

$$\Pi_0 \approx 0, \quad \Pi_{0i} \approx 0,\tag{31}$$

$$\tilde{\Pi}_0 \approx 0,\tag{32}$$

$$\Omega_i = (\Pi_i + \frac{1}{4m} \epsilon_{0ijk} H^{jk}) \approx 0, \quad \Lambda_{ij} = (\Pi_{ij} + \frac{1}{2m} \epsilon_{0ijk} G^k) \approx 0,\tag{33}$$

and the Hamiltonian is,

$$\begin{aligned}H_{St} &= \frac{1}{2}(\Pi_\theta)^2 - \frac{1}{2}\tilde{\Pi}_i \tilde{\Pi}^i + \frac{1}{4}(H_{ij} - B_{ij})(H^{ij} - B^{ij}) - \frac{1}{2}(G_i + \partial_i \theta)(G^i + \partial^i \theta) \\ &+ H^{0i}(\tilde{\Pi}_i - \frac{1}{m} \epsilon_{0ijk} \partial^j G^k) - G_0(\Pi_\theta + \frac{1}{2m} \epsilon_{0ijk} \partial^i H^{jk}) - B_0(\partial^i \tilde{\Pi}_i),\end{aligned}\tag{34}$$

and the Gauss law constraints are,

$$\Lambda = (\Pi_\theta + \frac{1}{2m} \epsilon_{0ijk} \partial^i H^{jk}) \approx 0,\tag{35}$$

$$\Lambda_i = -(\tilde{\Pi}_i - \frac{1}{m} \epsilon_{0ijk} \partial^j G^k) \approx 0,\tag{36}$$

$$\omega = \partial^i \tilde{\Pi}_i \approx 0.\tag{37}$$

Here Π_μ , $\Pi_{\mu\nu}$, $\tilde{\Pi}_\mu$ and Π_θ are the conjugate momenta corresponding to G^μ , $H^{\mu\nu}$, B^μ , and θ . Imposing the symplectic conditions (33), results in the same modified bracket as in (8). All the above constraints are first class as one can verify from their Poisson bracket algebra. The reducibility of the model is evident from the fact that the constraints Λ_i and ω are linearly dependent ($\partial^i \Lambda_i + \omega = 0$). Thus we have nine linearly independent first class constraints and hence the model defined by (29), describes massive spin-one particles.

III. Hamiltonian Embedding vs Lagrangian Embedding

In this section we make a comparative analysis of the constraint structure of BFM embedded model and the one following from the Stückelberg formulation (29). Note the following differences in the constraint structure of the Hamiltonian and Lagrangian embedded versions of this model (2).

1. The Hamiltonian embedding procedure modifies all the second class constraints of the original model whereas it is observed that the Stückelberg formulation modifies only the Gauss law constraints and not the primary constraints (This feature is present in the case Proca model also [5]).
2. The Hamiltonian embedded model has only *irreducible* first class constraints while the constraints of the Lagrangian embedded model obeys a *reducible* gauge algebra.

The first apparent difference between these two embeddings can be accounted by rewriting the Lagrangian (29) by dropping surface terms (i.e, by re expressing $\frac{1}{2}(H_{\mu\nu}B^{\mu\nu})$ as $-(\partial^\mu H_{\mu\nu}B^\nu)$ and $G_\mu\partial^\mu\theta$ as $-(\partial^\mu G_\mu\theta)$). With the Lagrangian written in this form, we get both the primary and the secondary constraints of the Stückelberg formulation which are structurally modified. Then the constraints will be in the same footing as the BFV embedded system.

Next, the second aspect, the difference in the reducible nature is clarified by studying the equivalence of the BFV embedded model to the Stückelberg formulation using the phase space path integral approach. For this, we start with the BFV embedded model partition function,

$$Z = \int D\eta \exp i \int d^4x L, \quad (38)$$

where the measure is,

$$D\eta = D\Pi_0 DG_0 DG_i D\Pi_{0i} DH_{0i} DH_{ij} D\Pi_\theta D\theta D\tilde{\Pi}_i DB_i D\lambda_\alpha D\Pi_\lambda^\alpha DP_\alpha DC_\alpha Dg_\alpha D\bar{g}_\alpha$$

and the $L = \mathcal{P}\partial^0\mathcal{Q} - \mathcal{H}_u$, \mathcal{P} is the generic momentum, and \mathcal{Q} is the generic field. Here \mathcal{H}_U is the unitarising Hamiltonian given in (28).

To show this equivalence, we chose the gauge fixing conditions to be

$$\delta(\chi_1) = \delta(G_0), \quad \delta(\chi_2) = \delta(H_{0i}), \quad (39)$$

$$\delta(\chi_3) = \delta(\partial_i G^i), \quad \delta(\chi_4^{ij}) = \delta(H^{ij} - \frac{1}{m}\epsilon^{0ijk}\partial_k G_0). \quad (40)$$

With this choice of χ_α , we get

$$\frac{1}{i\hbar}\{\Psi, Q\} = -\lambda_\alpha T^\alpha + P_\alpha g^\alpha - \chi_\alpha \Pi_\lambda^\alpha + \bar{g}_\alpha C^\beta \{\chi^\alpha, T_\beta\}. \quad (41)$$

The integrations over \bar{g}_α , and C_α give the Faddeev-Popov determinant $\det|\{\chi^\alpha, T_\beta\}|$, which is trivial in our case and we drop it hereafter. The integrations over g_α , and P_α give only constant numerical factors (which we omit in the following) and that of Π_λ^α and λ_α gives $\delta(T_\alpha)$ and $\delta(\chi_\alpha)$ respectively.

Thus, the partition function becomes

$$Z = \int D\Pi_0 DG_0 DG_i D\Pi_{0i} DH_{0i} DH_{ij} D\Pi_\theta D\theta D\tilde{\Pi}_i DB_i \delta(T_\alpha) \delta(\chi_\alpha) \exp i \int d^4x \mathcal{L}, \quad (42)$$

where, T_α are the first class constraints (22), and

$$\begin{aligned} \mathcal{L} = & \Pi_0 \partial^0 G^0 + \Pi_{0i} \partial^0 H^{0i} + \tilde{\Pi}_i \partial^0 B^i + \Pi_0 \partial^0 \theta \\ & + \frac{1}{4m} \epsilon_{0ijk} H^{ij} \partial^0 G^k - \frac{1}{4m} \epsilon_{0ijk} G^i \partial^0 H^{jk} - H. \end{aligned} \quad (43)$$

Here,

$$H = \mathcal{H} + (\Lambda + \Pi_\theta) \Pi_\theta + (\Lambda_i - \tilde{\Pi}_i) \tilde{\Pi}^i. \quad (44)$$

where \mathcal{H} is the gauge invariant Hamiltonian (20) and H differs from it only by terms proportional to the first class constraints and hence both these Hamiltonians define the same gauge system (i.e., on the constraint surface $H = \mathcal{H}$).

After exponentiating the constraints $\delta(\Lambda + \Pi_\theta)$ and $\delta(\Lambda_i - \tilde{\Pi}_i)$ using the Fourier transform fields c and d_i respectively, we integrate out Π_0 and Π_{0i} which are trivial due to the presence of the constraints $\delta(\Pi_0 + \theta)$ and $\delta(\Pi_{0i} - \tilde{\Pi}_i)$. Similarly the G_0 and H_{0i} integrations are trivial because of the gauge fixing conditions (40). Redefining c and d_i as G_0 and H_{0i} after integrating out Π_θ and $\tilde{\Pi}_i$, and using $\delta(\chi_3)$ and the condition $\partial^i H_{ij} = 0$ implied by $\delta(\chi_4^{ij})$, the partition function becomes

$$Z = \int DG_\mu DH_{\mu\nu} D\theta DB_i \delta(\chi_3) \delta(\chi_4^{ij}) \exp i \int d^4x \mathcal{L}, \quad (45)$$

where

$$\begin{aligned} \mathcal{L} = & \frac{1}{4m} \epsilon_{\mu\nu\lambda\sigma} H^{\mu\nu} G^{\lambda\sigma} + \frac{1}{2} (G_\mu + \partial_\mu \theta) (G^\mu + \partial^\mu \theta) - \frac{1}{4} (H_{ij} - B_{ij}) (H^{ij} - B^{ij}) \\ & - \frac{1}{2} H_{0i} H^{0i} + H_{0i} \partial^0 B^i - \frac{1}{2} (\partial^0 B_i) (\partial_0 B^i). \end{aligned} \quad (46)$$

It should be noted that the Lagrangian (46) is not manifestly covariant. This is in contrast with the other models like self-dual model in $2 + 1$ dimensions, Proca model in $3 + 1$ dimensions where the embedding procedure has been applied, where using the phase space path integral approach the corresponding Stückelberg Lagrangians have been obtained [5] in a manifestly covariant way. It should be noted here that B_0 , the time component of the vector Stückelberg field B_μ does not appear in the embedded model. To obtain the manifestly covariant Stückelberg Lagrangian (29), we further enlarge the phase space of the embedded model by introducing a pair of conjugate variables $\tilde{\Pi}_0$ and B_0 along with certain constraints such that the degrees of freedom is not changed. A natural choice is to introduce $\tilde{\Pi}_0 = 0$ as a first class constraint and $B_0 = 0$ as the corresponding gauge fixing condition and these conditions will remove the two extra degrees of freedom we have introduced (B_0 and $\tilde{\Pi}_0$). Thus the Gauss law constraint generated by demanding the consistency of $\tilde{\Pi}_0 \approx 0$ has to be linearly dependent with

the other first class constraints. We include the linearly independent constraint $\delta(\tilde{\Pi}_0)$ and the gauge fixing condition $\delta(B_0)$ in the path integral measure and $B_0\partial_i(\tilde{\Pi}_i + H_{0i})$ in the exponential along with the integrations over B_0 and $\tilde{\Pi}_i$ in eqn (42). Here the term $B_0\partial_i(\tilde{\Pi}_i + H_{0i})$ introduced in the path integral, will give the Gauss law constraint when we demand the persistency of $\tilde{\Pi}_0$, and this turn out to be a first class constraint which is linearly dependent with already existing first class constraint T_4^i (22) (*viz* : $\partial_i T_4^i + \partial_i(\tilde{\Pi}_i + H_{0i}) = 0$). Next using the constraint $\delta(\tilde{\Pi}_0)$ we carry out the integration over $\tilde{\Pi}_0$ dropping all $\tilde{\Pi}_0$ dependent terms. With the gauge condition $\delta(B_0)$ and integration over B_0 in the partition function, we get the Lagrangian in the exponential which is same as (29), showing the gauge equivalence of BFV embedded model to the Stückelberg formulation. Thus we see that the enlargement of the phase space of BFV embedded model is essential in obtaining the manifestly covariant Stückelberg action and it also brings out the reducible nature of the constraints.

In the earlier examples where the embedding had been applied [5,8,11–13], the number of the new phase space variables used to modify the original second class constraints to first class and the number of the phase space variables corresponding to the Stückelberg fields were the same. But in the case of the model studied in this paper, the embedded model has two phase space variables less than that of the Stückelberg formulation. It is interesting to note that this extra two variables and the associated constraints required to obtain the manifestly covariant Stückelberg Lagrangian starting from the phase space partition function of the embedded model also make the theory reducible. This seems to indicate a deep link between the manifest covariance of the Lagrangian and the reducible gauge structure.

IV. Conclusion

In this paper, we have made a comparative analysis of Hamiltonian and Lagrangian embedding of a first-order formulation of massive spin-one theory. Following the generalised Hamiltonian procedure of BFV, we have converted a second class system to a first class one and the nil-potent BFV-BRST charge and unitarising Hamiltonian were constructed. The canonical analysis of the gauge invariant Stückelberg Lagrangian corresponding to the initial gauge non-invariant theory was presented. We have then studied the relation between the Hamiltonian and Lagrangian embedding procedures. We find the first class constraints obtained by Hamiltonian embedding are irreducible but the ones obtained from Lagrangian embedding are reducible. Also the Lagrangian obtained from the BFV-Hamiltonian using the phase space path integral by integrating out all the momenta variables is not manifestly covariant. By further enlarging the BFV phase space by introducing new canonically conjugate variables with new constraints, manifestly covariant Stückelberg Lagrangian can be obtained starting from the phase space partition function of the embedded model. It turns out that one of the new constraints (Gauss law constraint) is linearly dependent on already existing constraints leading to reducibility. Thus this explains the difference in the reducible nature of the constraints in the Hamiltonian formulation of the Stückelberg theory and embedded model. It also

seems to imply a relationship between obtaining manifest covariance and reducibility of the constraints. A similar feature was also observed in a dually symmetric Maxwell theory where reducibility (of second class constraints) occur in the manifestly covariant formulation [14].

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